Glass synchronization in the network of oscillators with random phase shifts

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We investigate the synchronization properties in a network of oscillators with random phase shifts. In the absence of the shift, the system reduces to the well-known Kuramoto model, which displays synchronization of phases. We introduce an additional order parameter describing glass synchronization and obtain self-consistency equations through the use of the replica method. In the presence of appropriate phase shifts, the system is found to exhibit both phase synchronization and glass synchronization. It is also pointed out that the proper scaling of the coupling strength with the system size depends on the degree of randomness.
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I. INTRODUCTION

Large systems of mutually interacting oscillators have been used to explain the cooperative phenomena that prevail in physics, chemistry, biology, or even in social sciences. One of the most remarkable features of such an oscillator system is collective synchronization among its constituents, which is well understood by means of the Kuramoto phase model [1]. The model is simple and mathematically tractable because of its global coupling structure. This all-to-all global coupling, which might seem to be unrealistic at first sight, arises naturally in certain cases, e.g., intracavity lasers, series arrays of Josephson junctions, and biological systems [2,3]. Recently, it has been shown that a current-driven network of superconducting wires constitutes a physical realization of the Kuramoto model and interesting features of the network may be explained in terms of synchronization [4]. The network system has also attracted interest with regard to the possibility of glassiness in the presence of a transverse magnetic field [5–8], which is characterized by the divergence of the relaxation time, hysteresis, memory and aging effects, and an extensive number of metastable states separated by barriers scaling with the system size. In the context of synchronization, however, little is known about the glassy phase, which is in general believed to be caused by randomness and frustration. In existing studies of randomly interacting oscillators, neither clear evidence of glassiness nor the behavior of Edwards-Anderson (EA) order parameter has been reported [9,10].

The purpose of this paper is to investigate the effects of random phase shifts on synchronization properties in the network of oscillators. We introduce the EA order parameter and inspect the possibility of glass transition. Here also arises an interesting question as to the scaling of the coupling constants with the system size. In the Kuramoto model, each oscillator interacts with all the others, resulting in that the coupling constant should scale with the inverse of the system size N. On the other hand, in the wire network, the phase shift A_{ij} due to a strong magnetic field in the gauge-invariant phase difference takes rapidly fluctuating values and can be safely treated as a quenched random variable [4,6]. In this case, the proper scaling of the coupling constant goes with $1/\sqrt{N}$ in the thermodynamic limit. Note that the two models can be connected via the generalized model, where A_{ij}'s vary in the range $[-\gamma\pi,\gamma\pi)$ with a real number γ between 0 and 1. It is thus expected that the proper scaling of the coupling changes as γ is increased from zero to unity. It is revealed that the scaling changes rather abruptly at the fully random value $\gamma=1$.

This paper consists of four sections. In Sec. II we derive the effective Hamiltonian from the Fokker-Planck equation and obtain the self-consistency equations for the phase synchronization (PS) and the glass synchronization (GS) order parameters. It is shown that the glass transition appears only near the fully random region $\gamma\sim 1$. We also obtain schematic phase boundaries as functions of the coupling strength, noise strength, and randomness. Section III is devoted to the analysis of the network of superconducting wires and Sec. IV summarizes the main results.

II. EFFECTIVE HAMILTONIAN AND SELF-CONSISTENCY EQUATIONS

We consider a network of oscillators, each of which is characterized by its phase and coupled to all the other oscillators with strength K. The system is described by the set of equations of motion

$$\frac{d\theta_i}{dt}=\omega_i-K\sum_j \sin(\theta_i-\theta_j-A_{ij})+\xi_i(t), \tag{1}$$

where θ_i and ω_i are the phase and the natural frequency of the ith oscillator, respectively, and A_{ij}'s are random phase shifts taking values in the interval $[-\gamma\pi,\gamma\pi)$ with γ measuring the degree of randomness. The random noises $\xi_i(t)$ are characterized by

$$\langle\xi_i(t)\rangle=0,$$

$$\langle\xi_i(t)\xi_j(t)\rangle=2T\delta_{ij}\delta(t-t'),$$

where the noise strength T takes the role of the temperature. Without loss of generality, we set the mean natural frequency equal to zero and assume that ω_i's are distributed according to a Gaussian distribution with unit variance, which is achieved by rescaling time and measuring the coupling constant and the noise strength in units of the variance of the frequency distribution.

A convenient way to deal with a set of Langevin equations (1) is to resort to the Fokker-Planck equation for the appropriate probability density. In the Kuramoto model, it is convenient to introduce the order parameter $r\exp(i\phi) \equiv (1/N)\Sigma_j \exp(i\theta_j)$, which allows one to decouple the set of equations (1) and to deal with the single-oscillator probability density. The stationary solution of the corresponding Fokker-Planck equation for the one-oscillator probability density has been given at finite temperatures [11,12].

In the presence of the phase shifts, however, the resulting frustration prevents such reduction; we thus resort to the equation for the *N-oscillator* probability density $P(\{\theta_i\},t)$, which leads to a stationary solution $P^{(0)}(\{\theta_i\}) \propto \exp[-H(\{\theta_i\})/T]$ with the effective Hamiltonian

$$H = -\frac{K}{2}\sum_{i\neq j} \cos(\theta_i - \theta_j - A_{ij}) - \sum_i \omega_i \theta_i \tag{2}$$

at temperature T. In principle, the proper solution of the Fokker-Planck equation should satisfy the normalization and periodicity conditions, whereas in the above action the range of θ is extended and periodicity is lacking. Nevertheless, the inherent periodicity of the equation of motion allows one to treat the phase as an extended variable simply by taking the integration range from $-n\pi$ to $n\pi$. We thus regard H in Eq. (2) as the effective Hamiltonian with period $2n\pi$, where the limit $n\to\infty$ is to be taken. The validity of the above Hamiltonian with the extended variable can easily be justified, e.g., by means of the Villain approximation, which gives accurate results at low temperatures. Within the Villain scheme, it is straightforward to show that the corresponding action provides a stationary solution to the periodic Villain form of the Fokker-Planck equation and yields results independent of n. As we shall see below, the self-consistency equations for order parameters obtained from Eq. (2) are indeed independent of n and reproduce precisely all the known results in the appropriate limits. We thus believe that the Hamiltonian (2) describes the correct stationary properties of the system governed by Eq. (1) in the region of interest [13]. The average over the quenched random phase shifts $\{A_{ij}\}$ is performed through the use of the replica method. The replicated partition sum $\overline{Z^n}$ can be represented by

$$\overline{Z^n} \equiv \overline{\left\langle\!\!\left\langle \sum_{\{\theta^\alpha\}} e^{-H(\theta^\alpha)/T} \right\rangle\!\!\right\rangle_\omega}$$
$$= \sum_{\{\theta^\alpha\}} Z_1 Z_2 \left\langle\!\!\left\langle \exp\left((1/T)\sum_{\alpha,i} \omega_i \theta_i^\alpha \right) \right\rangle\!\!\right\rangle_\omega, \tag{3}$$

with

$$Z_1 \equiv \exp\left\{ \sum_\alpha \frac{aK}{2T}\left[\left(\sum_i \cos\theta_i^\alpha \right)^2 + \left(\sum_i \sin\theta_i^\alpha \right)^2 \right] \right\},$$

$$Z_2 \equiv \exp\left\{ \sum_{\alpha<\beta} \frac{1}{4}\left(\frac{K}{T}\right)^2 \sum_{i,j} [(1-2a^2+b)\cos(\theta_i^\alpha - \theta_j^\alpha) \times\cos(\theta_i^\beta - \theta_j^\beta) + (1-b)\sin(\theta_i^\alpha - \theta_j^\alpha)\sin(\theta_i^\beta - \theta_j^\beta)] \right\}$$
$$\otimes \exp\left\{ \sum_\alpha \frac{1}{8}\left(\frac{K}{T}\right)^2 \sum_{i,j} [(1-2a^2+b)\cos^2(\theta_i^\alpha - \theta_j^\alpha) + (1-b)\sin^2(\theta_i^\alpha - \theta_j^\alpha)] \right\}, \tag{4}$$

where $a \equiv \sin\gamma\pi/\gamma\pi$, $b \equiv \sin 2\gamma\pi/2\gamma\pi$, α and β are replica indices, and the overbar and $\langle\langle\ \rangle\rangle_\omega$ denote the quenched averages over the distribution of A_{ij} and over that of ω_i, respectively.

It is easy to check that without randomness ($\gamma=0$), $\ln Z_2$ vanishes and the Kuramoto model is recovered. At finite temperatures, the Kuramoto model has been analyzed through the use of the Fokker-Planck equation for the one-oscillator probability density [11,12]. Here we briefly show how the same results are reproduced by the replica method. The free energy per oscillator is given by

$$f = \lim_{n\to 0} \frac{T}{nN}\left[1 - \int \prod dA_\alpha dB_\alpha e^{-N\Phi} \right], \tag{5}$$

where

$$\Phi \equiv \frac{KN}{2T}\sum_\alpha (A_\alpha^2 + B_\alpha^2) - \ln \mathrm{Tr}_{[\theta]} \left\langle\!\!\left\langle \exp\left[\frac{KN}{T}\sum_\alpha (A_\alpha \cos\theta^\alpha + B_\alpha \sin\theta^\alpha) + \frac{1}{T}\sum_\alpha \omega\theta^\alpha \right] \right\rangle\!\!\right\rangle_\omega,$$

with the proper scaling of the coupling constant given by $K = \widetilde{K}/N$. In the limit $N\to\infty$, the mean-field equations follow from the saddle-point conditions and take the form

$$A_\alpha = \langle\!\langle \langle \cos\theta^\alpha \rangle \rangle\!\rangle_\omega,$$
$$B_\alpha = \langle\!\langle \langle \sin\theta^\alpha \rangle \rangle\!\rangle_\omega, \tag{6}$$

where $\langle\ \rangle$ stands for the average with respect to the action

$$\mathcal{L} = \exp\left[\frac{\widetilde{K}}{T}\sum_\alpha (A_\alpha \cos\theta^\alpha + B_\alpha \sin\theta^\alpha) + \frac{1}{T}\sum_\alpha \omega\theta^\alpha \right].$$

We adopt the replica-symmetric ansatz and set $A_\alpha = m_1$ and $B_\alpha = m_2$. Performing the average with respect to the Gaussian distribution of ω, we obtain the free energy in the form

$$f = \frac{\widetilde{K}}{2}(m_1^2 + m_2^2) - T\int Dz\ \ln I(x;z), \tag{7}$$

with $I(x;z) \equiv \Sigma_n (-1)^n I_n(x)[(z/T)^2 + n^2]^{-1}$ and $x \equiv (\widetilde{K}/T)\sqrt{m_1^2 + m_2^2}$, where I_n is the modified Bessel function of the first kind and $\int Dz$ represents the average over the normalized Gaussian variable z. The self-consistency equa-

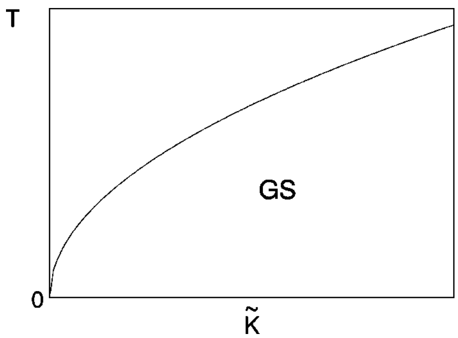

FIG. 1. Phase boundary for $\gamma=1$ on the $(\tilde{K},T)$ plane, below which glass synchronization appears.

tion for the PS order parameter $m\equiv\sqrt{m_1^2+m_2^2}$ is then obtained by differentiating the free energy with respect to m:

$$m=\int Dz\frac{I'(x;z)}{I(x;z)}, \tag{8}$$

where $I'(x;z)\equiv dI(x;z)/dx$. Here the shift of the integration range of θ in performing the trace in Eq. (5) does not change the final form of Eq. (8). In this sense, the correct periodicity is restored, which suggests that the Hamiltonian in Eq. (2) indeed describes the correct stationary behavior of the system in the whole region of interest. Note also that Eq. (8) indeed agrees with the result of Ref. [12] in the appropriate limit, displaying coherent motion characterized by global synchronization for $\tilde{K}$ exceeding the critical value $\tilde{K}_c$. In the zero-temperature limit, the direct expansion of Eq. (8) fails [4] and we adopt the spin-wave approximation, which is accurate at low temperatures. Analyzing Eq. (8) via the procedure largely similar to that in Ref. [4], one can easily obtain the familiar form

$$m=\tilde{K}m\int_{-1}^{1}\frac{dx}{\sqrt{2\pi}}e^{-(\tilde{K}mx)^2/2}\sqrt{1-x^2}, \tag{9}$$

which yields the critical value of $\tilde{K}$ at zero temperature: $\tilde{K}_c(T=0)=\sqrt{8/\pi}$. A typical phase diagram on the $(\tilde{K},T)$ plane has been shown in Fig. 1 of Ref. [4].

In the opposite limit of the fully random shift ($\gamma=1$), $\ln Z_1$ vanishes and the free energy per oscillator reads

$$f=\lim_{n\to 0}\frac{T}{nN}\left[1-\int\prod dQ_{\alpha\beta}dP_{\alpha\beta}e^{-N\Phi}\right], \tag{10}$$

with

$$\Phi\equiv\frac{N}{4}\left(\frac{K}{T}\right)^2\sum_{\alpha<\beta}[(Q_{\alpha\beta})^2+(P_{\alpha\beta})^2]-\ln \mathrm{Tr}_{[\theta]}\times\left\langle\!\!\left\langle\exp\left\{\frac{N}{2}\left(\frac{K}{T}\right)^2\sum_{\alpha<\beta}[Q_{\alpha\beta}\cos(\theta^\alpha-\theta^\beta)+P_{\alpha\beta}\sin(\theta^\alpha-\theta^\beta)]+\frac{1}{T}\sum_\alpha\omega\theta^\alpha\right\}\right\rangle\!\!\right\rangle_\omega,$$

which manifests that the coupling constant K should scale as $K=\tilde{K}/\sqrt{N}$. In the limit $N\to\infty$, the mean-field equations take the form

$$Q_{\alpha\beta}=\langle\!\langle\,\langle\cos(\theta^\alpha-\theta^\beta)\rangle\,\rangle\!\rangle_\omega,$$
$$P_{\alpha\beta}=\langle\!\langle\,\langle\sin(\theta^\alpha-\theta^\beta)\rangle\,\rangle\!\rangle_\omega, \tag{11}$$

with the action

$$\mathcal{L}=\exp\left\{\frac{1}{2}\left(\frac{\tilde{K}}{T}\right)^2\sum_{\alpha<\beta}[Q_{\alpha\beta}\cos(\theta^\alpha-\theta^\beta)+P_{\alpha\beta}\sin(\theta^\alpha-\theta^\beta)]+\frac{1}{T}\sum_\alpha\omega\theta^\alpha\right\}.$$

Note that Eq. (11) corresponds to $\langle\!\langle\,\langle s^{\alpha *}s^\beta\rangle\,\rangle\!\rangle_\omega$ in terms of the XY spin $s\equiv\exp(i\theta)$. Thus the order parameters $Q_{\alpha\beta}$ and $P_{\alpha\beta}$ are related to the EA order parameter measuring the overlaps between two replicas. Considering the U(1) symmetry of the whole system and assuming the replica symmetry, we set $Q_{\alpha\beta}=q$ and $P_{\alpha\beta}=0$, which yields the free energy in the form

$$f=\frac{\tilde{K}^2}{8T}(2q-q^2)-T\int DzDz_1Dz_2\ln I(x;z), \tag{12}$$

with $x\equiv\sqrt{q/2}(\tilde{K}/T)\sqrt{z_1^2+z_2^2}$. The self-consistency equation for the GS order parameter q is then given by

$$1-q=\int DzDz_1Dz_2\left\{\frac{I''(x;z)}{I(x;z)}-\left[\frac{I'(x;z)}{I(x;z)}\right]^2+\frac{I'(x;z)}{xI(x;z)}\right\}, \tag{13}$$

which allows a nontrivial solution for $\tilde{K}>\tilde{K}_g(T)$. Since the PS order parameter is always zero in this case ($\gamma=1$), the nontrivial solution $q\neq 0$ implies the appearance of the glass phase or GS. Adopting the spin-wave approximation in the zero-temperature limit, one can easily see that q approaches unity as $T\to 0$, which implies $\tilde{K}_g(T=0)=0$ regardless of γ. Figure 1 displays a schematic phase diagram for $\gamma=1$ on the $(\tilde{K},T)$ plane: Below the phase boundary $[\tilde{K}>\tilde{K}_g(T)]$, GS emerges.

We next consider the intermediate region $0<\gamma<1$. It is obvious that K is of the order of $N^{-\alpha}$ with $1/2\leq\alpha\leq 1$ in a globally coupled system, which makes Z_2 in Eq. (3) arbitrarily small compared to Z_1. This indicates that the randomness, unless it is perfect, does not destroy the PS but just renormalizes the coupling constant from K to aK. We thus conclude that the qualitative character of the system is not affected by a small amount of disorder and the behavior of the system changes rather abruptly at the fully random limit. To observe the transition more precisely, we set $\gamma=1-\kappa/\sqrt{N}$ together with $K\equiv\tilde{K}/\sqrt{N}$, which makes Z_1 and Z_2 be the same order. The free energy is then given by

$$f=\lim_{n\to 0}\frac{T}{nN}\left[1-\int\prod dA_\alpha dB_\alpha dQ_{\alpha\beta}dP_{\alpha\beta}e^{-N\Phi}\right], \tag{14}$$

with

$$\Phi \equiv \frac{\tilde{K}\kappa}{2T}\sum_{\alpha}(A_{\alpha}^2+B_{\alpha}^2)+\frac{1}{4}\left(\frac{\tilde{K}}{T}\right)^2\sum_{\alpha<\beta}[(Q_{\alpha\beta})^2+(P_{\alpha\beta})^2]$$

$$-\ln \mathrm{Tr}\left\langle\!\!\left\langle \exp\left\{\frac{1}{T}\sum_{\alpha}\omega\theta^{\alpha}+\frac{\tilde{K}\kappa}{T}\sum_{\alpha}(A_{\alpha}\cos\theta^{\alpha}+B_{\alpha}\sin\theta^{\alpha})+\frac{1}{2}\left(\frac{\tilde{K}}{T}\right)^2\sum_{\alpha<\beta}[Q_{\alpha\beta}\cos(\theta^{\alpha}-\theta^{\beta})+P_{\alpha\beta}\sin(\theta^{\alpha}-\theta^{\beta})]\right\}\right\rangle\!\!\right\rangle_{\omega}.$$

The saddle-point equations read

$$A_{\alpha}=\langle\!\langle \langle \cos\theta^{\alpha}\rangle \rangle\!\rangle_{\omega},$$

$$B_{\alpha}=\langle\!\langle \langle \sin\theta^{\alpha}\rangle \rangle\!\rangle_{\omega},$$

$$Q_{\alpha\beta}=\langle\!\langle \langle \cos(\theta^{\alpha}-\theta^{\beta})\rangle \rangle\!\rangle_{\omega},$$

$$P_{\alpha\beta}=\langle\!\langle \langle \sin(\theta^{\alpha}-\theta^{\beta})\rangle \rangle\!\rangle_{\omega},$$

with the corresponding action. We again assume the replica symmetry and set $A_{\alpha}=m_1$, $B_{\alpha}=m_2$, $Q_{\alpha\beta}=q$, and $P_{\alpha\beta}=0$, which gives the free energy in the form

$$f=\frac{\tilde{K}\kappa}{2}m^2+\frac{\tilde{K}^2}{8T}(2q-q^2)-T\int DzDz_1Dz_2\ln I(x;z), \tag{15}$$

where $x\equiv\sqrt{x_1^2+x_2^2}$ with

$$x_1\equiv\frac{1}{T}\left(\tilde{K}\kappa m+\sqrt{\frac{q}{2}}\tilde{K}z_1\right),$$

$$x_2\equiv\sqrt{\frac{q}{2}}\frac{\tilde{K}z_2}{T},$$

and the PS order parameter $m\equiv\sqrt{m_1^2+m_2^2}$. Differentiating the above free energy with respect to m and q, we finally get the self-consistency equations

$$m=\int DzDz_1Dz_2\frac{x_1I'(x;z)}{xI(x;z)}, \tag{16}$$

$$1-q=\int DzDz_1Dz_2\left\{\frac{I''(x;z)}{I(x;z)}-\left[\frac{I'(x);z}{I(x;z)}\right]^2+\frac{I'(x;z)}{xI(x;z)}\right\}. \tag{17}$$

Note that Eq. (17) is of the same form as Eq. (13) and reduces to the latter when $\kappa=0$. Equations (16) and (17) may be solved numerically to yield the phase boundaries in the $(\kappa,\tilde{K},T)$ space separating the synchronized, desynchronized, and glass phases from each other. Figure 2 shows schematically the zero-temperature phase boundary on the $(\kappa,\tilde{K})$ plane: Above the boundary, Eq. (16) yields a nontrivial solution for m, while Eq. (17) always has a nontrivial solution for q at zero temperature. Therefore, the phase boundary in

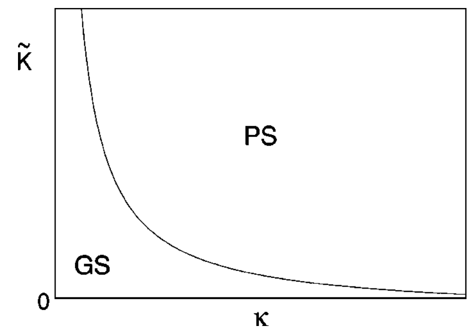

FIG. 2. Zero-temperature phase boundary on the $(\kappa,\tilde{K})$ plane, separating the (phase) synchronized state from the glass synchronized one.

Fig. 2 separates the phase state from the glass synchronized state. The schematic phase diagram at finite temperatures in the $(\kappa,\tilde{K},T)$ space is displayed in Fig. 3.

In the thermodynamic limit, however, κ is not a realizable parameter and it is natural to examine the behavior according to the value of γ. Unless $\gamma=1$, the PS order can be sustained and the proper scaling should be $K=\tilde{K}/N$ in the range $0\leq\gamma<1$. When $\tilde{K}$ exceeds the critical value $\tilde{K}_c$, the nontrivial solution for m exists, making Z_1 dominant. Here, although $\ln Z_2$ is of the order of $1/N$, q is still of the order of unity and determined by a function of nonzero m. In the case that $\tilde{K}<\tilde{K}_c$, the PS is destroyed ($m=0$) and the coupling constant in the self-consistency equation for the GS order parameter q becomes of the order of $1/\sqrt{N}$, which makes q also vanish in the thermodynamic limit. This indicates that the nonzero PS and GS order parameters appear at the same critical value $\tilde{K}_c$, which depends on the temperature. As γ approaches unity, $\tilde{K}_c$ grows arbitrarily large, as can be observed in Eq. (16), and finally at $\gamma=1$, the PS order is entirely banished. The region of the nonzero GS order parameter also shrinks as γ is increased; nevertheless, unlike the PS, the GS phase still survives at $\gamma=1$ and occupies a finite region, as shown in Fig. 1. In this fully random limit, the proper scaling becomes $K=\tilde{K}/\sqrt{N}$.

III. NETWORK OF SUPERCONDUCTING WIRES

In this section we consider a network of superconducting wires, which consists of two orthogonal sets of N parallel superconducting wires with Josephson junctions at each

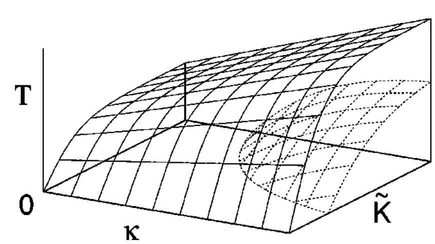

FIG. 3. Schematic phase diagram in $(\kappa,\tilde{K},T)$ space. Phase synchronization emerges below the surface drawn by dashed lines, whereas glass synchronization occurs between the two surfaces. Above the surface drawn by solid lines, neither of the two types of synchronization can exist.

node. Uniform current I^{ext} is injected at one edge of each wire and extracted at the opposite edge. The current conservation condition at each wire then gives the equations of motion [4]

$$\dot{\theta}_i^{(1)} = \frac{1}{N}\sum_j \dot{\theta}_j^{(2)} + \omega_i^{(1)} - K\sum_j \sin(\theta_i^{(1)} - \theta_j^{(2)}) + \xi_i(t),$$

$$\dot{\theta}_j^{(2)} = \frac{1}{N}\sum_i \dot{\theta}_i^{(1)} + \omega_j^{(2)} - K\sum_i \sin(\theta_j^{(2)} - \theta_i^{(1)}) + \xi_j(t), \tag{18}$$

where $\theta_i^{(1)}$ and $\theta_i^{(2)}$ are the phases of ith horizontal and vertical wires, respectively, $\omega_i \equiv 2eRI_i^{\text{ext}}/N\hbar$ corresponds to the natural frequency of the ith wire with the shunt resistance R, and K measures the coupling strength between the horizontal and vertical wires. As in the preceding section, one can obtain the Fokker-Planck equation for the $2N$-wire probability density and the resulting effective Hamiltonian. In the presence of a transverse magnetic field, the phase difference is replaced by the gauge-invariant one and the effective Hamiltonian takes the form

$$H = -K\sum_{i,j} \cos(\theta_i^{(1)} - \theta_j^{(2)} - A_{ij}) - \sum_i (\omega_i^{(1)}\theta_i^{(1)} + \omega_i^{(2)}\theta_i^{(2)}), \tag{19}$$

where $A_{ij} = (2\pi/\Phi_0)\int_i^j \mathbf{A}\cdot d\mathbf{l}$ with the flux quantum Φ_0. Adopting the Landau gauge $\mathbf{A} = Bx\mathbf{y}$ with B being the strength of applied field, we obtain $A_{ij} = 2\pi\phi ij/N$ with the flux per unit strip $\phi \equiv NBa^2/\Phi_0$, where a is the distance between two adjacent (parallel) wires. As is pointed out in Refs. [5,6], A_{ij}'s can be regarded as quenched random variables in the strong-field limit ($\phi \gg 1/N$). Further, it is natural to assume that A_{ij} and A_{ji} are mutually independent because the small difference in the areas made by the ith horizontal and jth vertical wires and by the jth horizontal and ith vertical wires leads to significantly large difference between A_{ij} and A_{ji}. This assumption allows quite an easy analysis because the correlations between the phases of horizontal wires and those of vertical wires vanish. Accordingly, the meaningful overlaps of horizontal and vertical wires are defined separately as

$$Q_{\alpha\beta}^{(1)} \equiv \langle\!\langle \langle \cos(\theta^{(1)\alpha} - \theta^{(1)\beta}) \rangle \rangle\!\rangle_{\omega^{(1)}},$$

$$Q_{\alpha\beta}^{(2)} \equiv \langle\!\langle \langle \cos(\theta^{(2)\alpha} - \theta^{(2)\beta}) \rangle \rangle\!\rangle_{\omega^{(2)}}. \tag{20}$$

It is then straightforward to write down the self-consistency equation for $q_k = Q_{\alpha\beta}^{(k)}$ ($k=1,2$),

$$1 - q_k = \int DzDz_1Dz_2 \left\{ \frac{I''(x_k;z)}{I(x_k;z)} - \left[\frac{I'(x_k;z)}{I(x_k;z)}\right]^2 + \frac{I'(x_k;z)}{x_k I(x_k;z)} \right\}, \tag{21}$$

with $x_k \equiv \sqrt{q_k/2}(\tilde{K}/T)\sqrt{z_1^2 + z_2^2}$, where q_1 and q_2 play the role of the EA order parameters of horizontal and vertical wires, respectively.

It is important to note here that the degree of randomness γ is not related to the strength of applied field in a direct way. Even in the weak-field limit, the distribution of the phase shift A_{ij} still covers the whole range $[-\pi,\pi)$, although the distribution is not random but periodic, depending on i and j. As a consequence, phase synchronization is destroyed by the existence of even a weak field, which is consistent with the existing studies on glassiness in the network of wires [5,6,8]. It is thus concluded that frustration rather than randomness plays an important role in glassiness. In fact, it has been shown that the randomness without frustration can be easily removed [10]. The quantitative investigation of the GS order parameter in the weak-field limit would be an interesting topic for further study.

IV. SUMMARY

We have studied the synchronization properties in the networks of coupled oscillators with random phase shifts, with emphasis on the possibility of glass synchronization. While the system without randomness is well known to display phase synchronization, the fully random system is characterized by glass synchronization. To investigate in detail the effects of the random phase shift, we have introduced a parameter controlling the degree of randomness and obtained the self-consistency equations for the phase synchronization and the glass synchronization order parameters through the use of the replica method. It is found that partial randomness in the phase shift merely alters the coupling constant, still leaving the phase synchronization possible. In this case, the proper scaling of the coupling constant goes as $1/N$ and the corresponding equations of motion under this scaling reveal that the nonzero phase and glass synchronization order parameters emerge at the same value of the coupling strength. This indicates that there does not exist the glassy phase in the system with partially random phase shifts, characterized by the nonzero glass synchronization order parameter together with the vanishing PS order parameter. Near the fully random regime, on the other hand, the phase synchronization order parameter diminishes to zero, but a finite region of the nonvanishing glass synchronization order parameter still remains, signaling the glass phase in the system. In this fully random case, the proper scaling for the glass transition is given by $1/\sqrt{N}$, as expected. We have thus obtained a schematic phase diagram in the three-dimensional space of the temperature, coupling strength, and randomness.

We have also considered the network of superconducting wires in the presence of positional disorder. In the strong-field limit, the bond angles (phase shifts) between the horizontal set and the vertical set of wires may be regarded as quenched random variables. Further, in the presence of the positional disorder, it is allowed to treat them as independent of each other. This leads to the self-consistency equations for the glass synchronization order parameters, which are largely similar in form to those in the globally coupled system. Thus the qualitative behavior is concluded to be the same, which originates from the fact that positional disorder tends to suppress the overlap between the two sets of wires.

Finally, we point out that the network of globally coupled oscillators, in which the static features of the glass transition have been studied here, may be regarded as the mean-field version of the XY gauge-glass model. The gauge-glass model has been extensively studied in relation to the vortex-glass phase of disordered superconductors [14]. In two dimensions, equilibrium studies have suggested the absence of glass order at finite temperatures [15], while the dynamical investigations seem to favor a finite-temperature glass transition, leading to controversy as to its nature as well as existence [16]. It should be noted that a dynamic glass transition is not necessarily accompanied by an equilibrium one and provides a wealth of interesting phenomena, which cannot be inferred from equilibrium studies. It is thus of interest to investigate the dynamics of the network system, which is left for future study.

ACKNOWLEDGMENTS

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